Solutions to short-answer questions

1 **a** $\text{Maximum} = 5, \text{minimum} = 1$

b $\text{Maximum} = 4, \text{minimum} = -2$

c $\text{Maximum} = 4, \text{minimum} = -4$

d $\text{Maximum} = 2, \text{minimum} = 0$

e $\text{Maximum} = 1 \text{ (when } \cos\theta = -1), \text{minimum} = \dfrac{1}{3}$

2 **a**
$$\begin{aligned} \tan^2\theta &= \frac{1}{3} \\ \tan\theta &= \pm\frac{1}{\sqrt{3}} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

b
$$\begin{aligned} \tan 2\theta &= -1 \\ 2\theta &= \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4} \\ \theta &= \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8} \end{aligned}$$

c
$$\begin{aligned} \sin 3\theta &= -1 \\ 3\theta &= \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2} \\ \theta &= \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

d
$$\begin{aligned} \cos 2\theta &= \frac{1}{\sqrt{2}} \\ 2\theta &= \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4} \\ \theta &= \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8} \end{aligned}$$

3 **a**
$$\begin{aligned} \sec\theta + \operatorname{cosec}\theta\cot\theta &= \frac{1}{\cos\theta} + \frac{1}{\sin\theta} \times \frac{\cos\theta}{\sin\theta} \\ &= \frac{1}{\cos\theta}\left(1 + \frac{\cos\theta}{\sin\theta} \times \frac{\cos\theta}{\sin\theta}\right) \\ &= \frac{1}{\cos\theta}(1 + \cot^2\theta) \\ &= \sec\theta\operatorname{cosec}^2\theta \end{aligned}$$

b
$$\begin{aligned} \frac{\tan^2\theta + \cos^2\theta}{\sec\theta + \sin\theta} &= \frac{\tan^2\theta + 1 - \sin^2\theta}{\sec\theta + \sin\theta} \\ &= \frac{\sec^2\theta - \sin^2\theta}{\sec\theta + \sin\theta} \\ &= \frac{(\sec\theta - \sin\theta)(\sec\theta + \sin\theta)}{\sec\theta + \sin\theta} \\ &= \sec\theta - \sin\theta \end{aligned}$$

4 $\cos^2 A = 1 - \sin^2 A$

$= 1 - \frac{25}{169} = \frac{144}{169}$

$\cos A = \frac{12}{13}$ (Since A is acute)

$\cos^2 B = 1 - \sin^2 B$

$= 1 - \frac{64}{289} = \frac{225}{289}$

$\cos B = \frac{15}{17}$ (Since B is acute)

a $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$= \frac{12}{13} \times \frac{15}{17} - \frac{5}{13} \times \frac{8}{17}$

$= \frac{140}{221}$

b $\sin(A + B) = \sin A \cos B - \cos A \sin B$

$= - \times \frac{5}{13} \times \frac{15}{17} - \frac{12}{13} \times \frac{8}{17}$

$= -\frac{21}{221}$

c $\tan A = \frac{\sin A}{\cos A} = \frac{5}{12}$

$\tan B = \frac{\sin B}{\cos B} = \frac{8}{15}$

$\tan(A + B) = \frac{\tan a + \tan B}{1 - \tan A \tan B}$

$= \left(\frac{5}{12} + \frac{8}{15}\right)$

$\div \left(1 - \frac{5}{12} \times \frac{8}{15}\right)$

$= \frac{57}{60} \div \frac{7}{9}$

$= \frac{19}{20} \times \frac{9}{7}$

$= \frac{171}{140}$

5 a $\text{Expression} = \cos(80^\circ - 20^\circ)$

$= \cos 60^\circ = \frac{1}{2}$

b $\text{Expression} = \tan(15^\circ + 30^\circ)$

$= \tan 45^\circ = 1$

6 a $\text{Expression} = \sin(A + B)$

$= \sin \frac{\pi}{2} = 1$

b $\text{Expression} = \cos(A + B)$

$= \cos \frac{\pi}{2} = 0$

7 a $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B$

$= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B$

$= \sin^2 A - \sin^2 B$

b Left side $= \dfrac{\sin^2\theta + (1+\cos\theta)^2}{\sin\theta(1+\cos\theta)}$

$$= \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{2 + 2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)}$$

$$= \frac{2}{\sin\theta}$$

c Left side $= \dfrac{\sin\theta(1 - 2\sin^2\theta)}{\cos\theta(2\cos^2\theta - 1)}$

$$= \frac{\sin\theta(1 - \sin^2\theta - \sin^2\theta)}{\cos\theta(\cos^2\theta + \cos^2\theta - 1)}$$

$$= \frac{\sin\theta(\cos^2\theta - \sin^2\theta)}{\cos\theta(\cos^2\theta - (1 - \cos^2\theta))}$$

$$= \frac{\sin\theta(\cos^2\theta - \sin^2\theta)}{\cos\theta(\cos^2\theta - \sin^2\theta)}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta$$

8 $\cos^2 A = 1 - \sin^2 A$

$$= 1 - \frac{5}{9} = \frac{4}{9}$$

$\cos A = -\dfrac{2}{3}$ (Since A is obtuse)

a $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{4}{9} - \frac{5}{9}$$

$$= -\frac{1}{9}$$

b $\sin 2A = 2\sin A\cos A$

$$= 2 \times \frac{\sqrt{5}}{3} \times -\frac{2}{3}$$

$$= -\frac{4\sqrt{5}}{9}$$

c $\sin 4A = 2\sin 2A\cos 2A$

$$= 2 \times -\frac{4\sqrt{5}}{9} \times -\frac{1}{9}$$

$$= \frac{8\sqrt{5}}{81}$$

9

a $\text{Left side} = \dfrac{1 - \dfrac{\sin^2\theta}{\cos^2\theta}}{1 + \dfrac{\sin^2\theta}{\cos^2\theta}}$

$$= \dfrac{1 - \dfrac{\sin^2\theta}{\cos^2\theta}}{1 + \dfrac{\sin^2\theta}{\cos^2\theta}} \times \dfrac{\cos^2\theta}{\cos^2\theta}$$

$$= \dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \dfrac{\cos 2\theta}{1} = \cos 2\theta$$

b $\text{Left side} = \dfrac{\sin^2 A + (1 + \cos A)^2}{\sin A(1 + \cos A)}$

$$= \dfrac{\sin^2 A + 1 + 2\cos A + \cos^2 A}{\sin A(1 + \cos A)}$$

$$= \dfrac{2 + 2\cos A}{\sin A(1 + \cos A)}$$

$$= \dfrac{2(1 + \cos A)}{\sin A(1 + \cos A)}$$

$$= \dfrac{2}{\sin A}$$

10a $\tan 15^\circ = \tan\ (60 - 45)^\circ$

$$= \dfrac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ}$$

$$= \dfrac{\sqrt{3} - 1}{1 + \sqrt{3}}$$

$$= \dfrac{\sqrt{3} - 1}{1 + \sqrt{3}} \times \dfrac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \dfrac{3 - 2\sqrt{3} + 1}{3 - 1}$$

$$= 2 - \sqrt{3}$$

b $\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

Add the two equations:

$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$

11a Express in the form $r\sin(x+\alpha)=1$.

$$r=\sqrt{1+1}=\sqrt{2}$$
$$\cos\alpha=\frac{1}{\sqrt{2}};\ \sin\alpha=\frac{1}{\sqrt{2}}$$
$$\alpha=\frac{\pi}{4}$$
$$\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)=1$$
$$\sin\left(x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$
$$x+\frac{\pi}{4}=\frac{\pi}{4},\frac{3\pi}{4},\frac{9\pi}{4}$$
$$x=0,\ \frac{\pi}{2},\ 2\pi$$

b
$$2\sin\frac{x}{2}\cos\frac{x}{2}=-\frac{1}{2}$$
$$\sin\left(2\times\frac{x}{2}\right)=-\frac{1}{2}$$
$$\sin x=-\frac{1}{2}$$
$$x=\frac{7\pi}{6},\frac{11\pi}{6}$$

c
$$3\times\frac{2\tan x}{1-\tan^2 x}=2\tan x$$
$$2\tan x\left(\frac{3}{1-\tan^2 x}-1\right)=0$$
$$2\tan x\left(\frac{3-(1-\tan^2 x)}{1-\tan^2 x}\right)=0$$
$$\tan x=0\ (\text{since} 2+\tan^2 x\neq 0)$$
$$x=0,\pi,2\pi$$

d
$$\sin^2 x-\cos^2 x=1$$
$$\cos 2x=-1$$
$$2x=\pi,3\pi$$
$$x=\frac{\pi}{2},\frac{3\pi}{2}$$

e
$$\sin(3x-x)=\frac{\sqrt{3}}{2}$$
$$\sin 2x=\frac{\sqrt{3}}{2}$$
$$2x=\frac{\pi}{3},\frac{2\pi}{3},\frac{7\pi}{3},\frac{8\pi}{3}$$
$$x=\frac{\pi}{6},\frac{\pi}{3},\frac{7\pi}{6},\frac{4\pi}{3}$$

f $\cos\left(2x - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$$2x - \frac{\pi}{3} = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}$$

$$2x = \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{19\pi}{6}, \frac{21\pi}{6}$$

$$x = \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{19\pi}{12}, \frac{7\pi}{4}$$

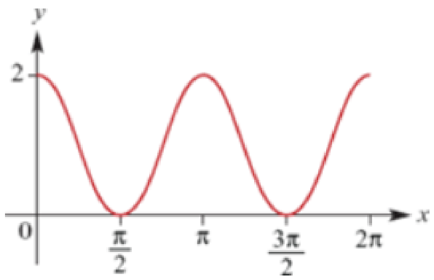
12a $y = 2\cos^2 x$

$$= \cos^2 x + (1 - \sin^2 x)$$

$$= \cos^2 x - \sin^2 x + 1$$

$$= \cos 2x + 1$$

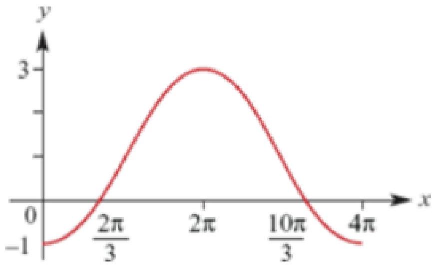
The graph of $y = \cos 2x$ (amplitude 1, period π) raised 1 unit.



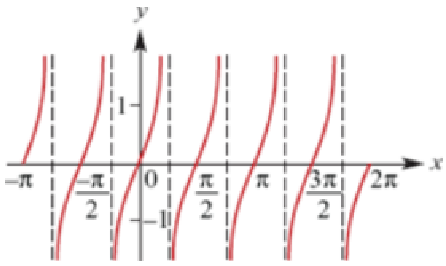
b The graph is

$$y = 1 - 2\sin\left(\frac{\pi}{2} - \frac{x}{2}\right) = 1 - 2\cos\frac{\pi}{2}.$$

It is $y = 2\cos\frac{x}{2}$ (period 4π) reflected in the x-axis and raised 1 unit.



c The normal tangent graph, but with period $\frac{\pi}{2}$.



13

$$\tan(\theta + A) = 4$$

$$\frac{\tan\theta + \tan A}{1 - \tan\theta\tan A} = 4$$

$$\frac{\tan\theta + 2}{1 - 2\tan\theta} = 4$$

$$\tan\theta + 2 = 4(1 - 2\tan\theta)$$

$$= 4 - 8\tan\theta$$

$$9\tan\theta = 2$$

$$\tan\theta = \frac{2}{9}$$

14a $r=\sqrt{4+81}=\sqrt{85}$

$\cos\alpha=\dfrac{2}{\sqrt{85}};\ \sin\alpha=\dfrac{9}{\sqrt{85}}$

$\sqrt{85}\cos(\theta-\alpha)$, where

$\alpha=\cos^{-1}\left(\dfrac{2}{\sqrt{85}}\right)$

b **i** $\sqrt{85}$

ii $\cos(\theta-\alpha)=1$

$$\begin{aligned}\theta-\alpha&=0\\ \theta&=-\alpha\\ \cos\theta&=\cos\alpha\\ &=\frac{2}{\sqrt{85}}\end{aligned}$$

iii Solve $\sqrt{85}\cos(\theta+\alpha)=1$.

$$\begin{aligned}\cos(\theta-\alpha)&=\frac{1}{\sqrt{85}}\\ \theta-\alpha&=\cos^{-1}\left(\frac{1}{\sqrt{85}}\right)\\ \theta&=\alpha+\cos^{-1}\left(\frac{1}{\sqrt{85}}\right)\\ &=\cos^{-1}\left(\frac{2}{\sqrt{85}}\right)\\ &\quad+\cos^{-1}\left(\frac{1}{\sqrt{85}}\right)\end{aligned}$$

15a $\sin4\theta+\sin2\theta=0$

$$\begin{aligned}2\sin3\theta\sin\theta&=0\\ \therefore\sin3\theta=0\text{ or }\sin\theta&=0\\ \theta&=0,\frac{\pi}{3},\frac{\pi}{2},\frac{2\pi}{3},\pi\end{aligned}$$

b $\sin2\theta-\sin\theta=0$

$$\begin{aligned}2\sin\frac{\theta}{2}\cos\frac{3\theta}{2}&=0\\ \therefore\sin\frac{\theta}{2}=0\text{ or }\cos\frac{3\theta}{2}&=0\\ \theta&=0,\frac{\pi}{3},\pi\end{aligned}$$

16 $\text{LHS} = \dfrac{\cos A - \cos B)}{\sin A + \sin B}$

$$= \frac{-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$$

$$= \frac{-\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)}$$

$$= \tan\left(\frac{B-A}{2}\right)$$

$$= \text{RHS}$$

Solutions to multiple-choice questions

1 **A**

$$\begin{aligned}\text{cosec}x - \sin x &= \frac{1}{\sin x} - \sin x\\ &= \frac{1-\sin^2 x}{\sin x}\\ &= \frac{\cos^2 x}{\sin x}\\ &= \cos x \times \frac{\cos x}{\sin x}\\ &= \cos x \cot x\end{aligned}$$

2 **A**

$$\begin{aligned}\cos x &= -\frac{1}{3}\\ \cos^2 x + \sin^2 x &= 1\\ \left(-\frac{1}{3}\right)^2 + \sin^2 x &= 1\\ \sin^2 x &= 1 - \frac{1}{9} = \frac{8}{9}\\ \sin x &= \pm\sqrt{\frac{8}{9}}\\ &= -\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}\end{aligned}$$

3 **B**

$$\begin{aligned}\sec\theta &= \frac{b}{a}\\ \tan^2\theta + 1 &= \sec^2\theta\\ \tan^2\theta &= \frac{b^2}{a^2} = 1\\ &= \frac{b^2 - a^2}{a^2}\\ \tan\theta &= \frac{\sqrt{b^2-a^2}}{a}\end{aligned}$$

(Since $\tan\theta > 0$)

4 **A** $\angle ABC = u; \angle XBC = v$

$$\tan u = \frac{x+4}{2};\ \tan v = \frac{x}{2}$$

$$\begin{aligned}\tan\theta &= \tan(u-v)\\ &= \frac{\tan u - \tan v}{1+\tan u \tan v}\\ &= \frac{\frac{x+4}{2}-\frac{x}{2}}{1+\frac{x+4}{2}\times\frac{x}{2}}\\ &= \frac{4}{2} \div \frac{4+x(x+4)}{4}\\ &= 2\times\frac{4}{x^2+4x+4}\\ &= \frac{8}{(x+2)^2}\end{aligned}$$

$$\begin{aligned}\sin^2 A &= 1-\cos^2 A\\ &= 1-t^2\end{aligned}$$

5 **C** $\sin A = \sqrt{1-t^2}$

(Since $\sin A > 0$)

$$\begin{aligned}\cos^2 B &= 1-\sin^2 B\\ &= 1-t^2\\ \cos B &= -\sqrt{1-t^2}\end{aligned}$$

Since $\cos B < 0$)

$$\begin{aligned}\sin(B+A) &= \sin B\cos A + \cos B \sin A\\ &= t\times t + \left(-\sqrt{1-t^2}\right)\times\sqrt{1-t^2}\\ &= t^2-(1-t^2)\\ &= 2t^2-1\end{aligned}$$

6 **E**

$$\begin{aligned}\frac{\sin 2A}{\cos 2A - 1} &= \frac{2\sin A\cos A}{\cos^2 A - \sin^2 A - 1}\\ &= \frac{2\sin A\cos A}{-\sin^2 A-(1-\cos^2 A)}\\ &= \frac{2\sin A\cos A}{-\sin^2 A - \sin^2 A}\\ &= \frac{2\sin A\cos A}{-2\sin^2 A}\\ &= \frac{\cos A}{\sin A}\\ &= -\cot A\end{aligned}$$

7 **E**

$$\begin{aligned}(1+\cot x)^2+(1-\cot x)^2 &= 1+2\cot x+\cot^2 x+1-2\cot x+\cot^2 x\\ &= 2+2\cot^2 x\\ &= 2(1+\cot^2 x)\\ &= 2\text{cosec}^2 x\end{aligned}$$

8 A $\sin 2A = 2\sin A\cos A$

$$m = 2\sin A \times n$$
$$\sin A = \frac{m}{2n}$$
$$\tan A = \frac{\sin A}{\cos A}$$
$$= \frac{m}{2n} \times \frac{1}{n}$$
$$= \frac{m}{2n^2}$$

9 D $r = \sqrt{1+1} = \sqrt{2}$

$$\cos\alpha = \frac{1}{\sqrt{2}};\ \sin\alpha = -\frac{1}{\sqrt{2}}$$

A positive angle must be chosen,

$$\therefore\ \alpha = \frac{7\pi}{4}$$
$$\sqrt{2}\sin\left(x + \frac{7\pi}{4}\right)$$

10 E

Solutions to extended-response questions

1 a $P = AD + DC + CB + BA$

$$= 2AO + BA + 2AO + BA$$
$$= 4AO + 2BA$$
$$= 4 \times 5\cos\theta + 2 \times 5\sin\theta$$
$$= 20\cos\theta + 10\sin\theta, \text{ as required.}$$

b

$$a = 20,\ b = 10 \text{ and } R = \sqrt{a^2 + b^2}$$
$$= \sqrt{20^2 + 10^2}$$
$$= \sqrt{500}$$
$$= 10\sqrt{5}$$
$$\text{Now } \cos\alpha = \frac{a}{R}$$
$$= \frac{20}{10\sqrt{5}}$$
$$= \frac{2}{\sqrt{5}}$$
$$= \frac{2\sqrt{5}}{5}$$
$$\text{Also } \sin\alpha = \frac{b}{R}$$
$$= \frac{10}{10\sqrt{5}}$$
$$= \frac{1}{\sqrt{5}}$$
$$= \frac{\sqrt{5}}{5}$$

$$\text{Hence, } 0 < \alpha < 90 \text{ and } \alpha^\circ = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)^\circ = (26.565\,05\ldots)^\circ$$
$$\text{Hence } P = R\cos(\theta - \alpha)$$
$$= 10\sqrt{5}\cos(\theta - \alpha) \text{ where } \alpha = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

When $P = 16$,

$$10\sqrt{5}\cos(\theta - \alpha) = 16$$

$$\therefore \quad \cos(\theta - \alpha) = \frac{16}{10\sqrt{5}}$$

$$\therefore \quad (\theta - \alpha)^\circ = \cos^{-1}\left(\frac{8}{5\sqrt{5}}\right)^\circ$$

$$\therefore \quad \theta^\circ = \cos^{-1}\left(\frac{8}{5\sqrt{5}}\right)^\circ + \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)^\circ$$

When $P = 16, \theta = 70.88^\circ$

c Area of rectangle $= AB \times AD$

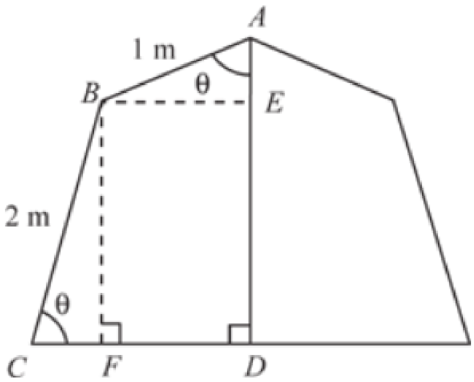
$$= 5\sin\theta \times 2AO$$
$$= 5\sin\theta \times 2 \times 5\cos\theta$$
$$= 50\sin\theta\cos\theta$$
$$= 25 \times 2\sin\theta\cos\theta$$
$$= 25\sin 2\theta$$
$$\therefore \quad k\sin 2\theta = 25\sin 2\theta$$
$$\therefore \quad k = 25$$

d Area is a maximum when $\sin 2\theta = 1$

$$\therefore \quad 2\theta = 90^\circ$$
$$\therefore \quad \theta = 45^\circ$$

2 a $AD = AE + ED$

$$= \cos\theta + BF$$
$$= \cos\theta + 2\sin\theta$$



b $a = 1,\ b = 2$ and $R = \sqrt{a^2 + b^2}$

$$= \sqrt{1^2 + 2^2}$$
$$= \sqrt{5}$$

Now $\cos\alpha = \dfrac{a}{R}$

$$= \frac{1}{\sqrt{5}}$$
$$= \frac{\sqrt{5}}{5}$$

Also $\sin\alpha = \dfrac{b}{R}$

$$= \frac{2}{\sqrt{5}}$$
$$= \frac{2\sqrt{5}}{5}$$

Hence, $0 < \alpha < 90$ and $\alpha^\circ = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)^\circ = (63.434\,94\ldots)^\circ$

Hence $AD = \sqrt{5}\cos(\theta - 63)^\circ$

c The maximum length of *AD* is $\sqrt{5}$ metres.
When $AD = \sqrt{5}$,

$$\sqrt{5}\cos(\theta - 63)^\circ = \sqrt{5}$$
$$\therefore \quad \cos(\theta - 63)^\circ = 1$$
$$\therefore \quad \theta - 63 = 0$$
$$\therefore \quad \theta = 63$$

d When $AD = 2.15$,

$$\sqrt{5}\cos(\theta - \alpha)^\circ = 2.15$$
$$\therefore \quad \cos(\theta - \alpha)^\circ = \frac{2.15}{\sqrt{5}}$$
$$\therefore \quad (\theta - \alpha)^\circ = \cos^{-1}\left(\frac{2.15}{\sqrt{5}}\right)^\circ$$
$$= (15.948\,46\ldots)^\circ$$
$$\therefore \quad \theta = (15.948 + 63.435)^\circ$$

The value of θ, for which $\theta > \alpha$, is 79.38°.

3 a

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{1 - \tan^2\theta}{\sec^2\theta}$$
$$= \cos^2\theta\,(1 - \tan^2\theta)$$
$$= \cos^2\theta - \frac{\cos^2\theta\sin^2\theta}{\cos^2\theta}$$
$$= \cos^2\theta - \sin^2\theta$$

Hence, $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$, as required.

b

i $$\text{From } \mathbf{a}, \cos\left(2 \times 67\frac{1}{2}^\circ\right) = \frac{1 - \tan^2\left(67\frac{1}{2}^\circ\right)}{1 + \tan^2\left(67\frac{1}{2}^\circ\right)}$$

$$\therefore \quad \cos 135^\circ = \frac{1 - x^2}{1 + x^2} \text{ where } x = \tan\left(67\frac{1}{2}^\circ\right)$$
$$\therefore \quad -\cos 45^\circ = \frac{1 - x^2}{1 + x^2}$$
$$\therefore \quad -\frac{1}{\sqrt{2}} = \frac{1 - x^2}{1 + x^2}$$
$$\therefore \quad -\sqrt{2} = \frac{1 + x^2}{1 - x^2}$$
$$\therefore \quad 1 + x^2 = -\sqrt{2}(1 - x^2)$$
$$\therefore \quad 1 + x^2 = \sqrt{2}x^2 - \sqrt{2}, \text{ as required.}$$

ii

$$1+x^2=\sqrt{2}x^2-\sqrt{2}$$
$$\therefore 1+\sqrt{2}=\sqrt{2}x^2-x^2$$
$$=x^2(\sqrt{2}-1)$$
$$\therefore \quad x^2=\frac{1+\sqrt{2}}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}$$
$$=\frac{\sqrt{2}+1+2+\sqrt{2}}{2-1}$$
$$=3+2\sqrt{2} \qquad \ldots \boxed{1}$$

Given $\tan\left(67\frac{1}{2}^\circ\right)=a+b\sqrt{2}$

$$\therefore \quad x=a+b\sqrt{2} \text{ where } x=\tan\left(67\frac{1}{2}^\circ\right)$$
$$\therefore \quad x^2=(a+b\sqrt{2})^2$$
$$=a^2+2\sqrt{2}ab+2b^2$$
$$=(a^2+2b^2)+(2ab)\sqrt{2} \qquad \ldots \boxed{2}$$

Equating $\boxed{1}$ and $\boxed{2}$

$$a^2+2b^2=3 \qquad \ldots \boxed{3}$$
$$2ab=2$$
$$ab=1$$

As a and b are integers, $a=1, b=1$ or $a=-1, b=-1$ and $(1+\sqrt{2})^2=3+2\sqrt{2}$

Note: An alternative method is to note

$$x^2=\frac{\sqrt{2}+1}{\sqrt{2}-1}$$
$$=\frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)(\sqrt{2}+1)}$$
$$=(\sqrt{2}+1)^2$$
$$\therefore x=\pm(\sqrt{2}+1)$$

When $b=-1, a=-1$,

$a+b\sqrt{2}=-1-\sqrt{2}$

When $b=1, a=1$,

$a+b\sqrt{2}=1+\sqrt{2}$

But $\tan\left(67\frac{1}{2}^\circ\right)>0$,

$$\therefore a+b\sqrt{2}=\sqrt{2}+1$$
$$=1+\sqrt{2}$$
$$\therefore \quad a=1,\ b=1$$

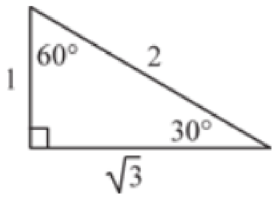
c $$\tan\left(7\frac{1}{2}^\circ\right) = \tan\left(67\frac{1}{2}^\circ - 60^\circ\right)$$

$$= \frac{\tan\left(67\frac{1}{2}^\circ\right) - \tan(60^\circ)}{1 + \tan\left(67\frac{1}{2}^\circ\right)\tan(60^\circ)}$$

\) $\tan 60^\circ = \sqrt{3}$\(

$$= \frac{1+\sqrt{2}-\sqrt{3}}{1+(1+\sqrt{2})\sqrt{3}}$$

$$= \frac{1+\sqrt{2}-\sqrt{3}}{1+\sqrt{3}+\sqrt{6}}$$



4 a i $$\angle CBA = \pi - \frac{2\pi}{5} = \frac{3\pi}{5}$$

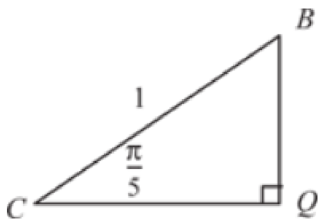
$$\angle BCA = \frac{1}{2}\left(\pi - \frac{3\pi}{5}\right) \text{ as} \angle BCA = \angle BAC (\triangle ABC \text{ is isosceles})$$

$$= \frac{1}{2} \times \frac{2\pi}{5} = \frac{\pi}{5}, \text{ as required.}$$

ii $$CA = 2CQ$$

$$= 2\cos\frac{\pi}{5}$$

The length of CA is $2\cos\frac{\pi}{5}$ units.



b i $$\angle DCP = \angle BCD - \angle BCA$$

$$= \angle CBA - \angle BCA$$

$$= \frac{3\pi}{5} - \frac{\pi}{5} = \frac{2\pi}{5}, \text{ as required.}$$

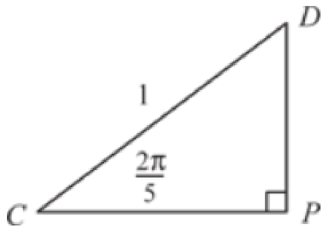
ii $$AC = 2CP + PR$$

$$= 2\cos\frac{2\pi}{5} + DE$$

$$= 2\cos\frac{2\pi}{5} + 1$$

But $AC = 2\cos\frac{\pi}{5}$ (from **a ii**)

$\therefore \quad 2\cos\frac{\pi}{5} = 2\cos\frac{2\pi}{5} + 1$, as required.



iii

$$2\cos\frac{\pi}{5} = 2\cos\frac{2\pi}{5} + 1$$

$$\therefore \quad 2\cos\frac{2\pi}{5} = 2\cos\frac{\pi}{5} - 1$$

$$\therefore \quad \cos\frac{2\pi}{5} = \cos\frac{\pi}{5} - \frac{1}{2}$$

$$\therefore \quad 2\cos^2\frac{\pi}{5} - 1 = \cos\frac{\pi}{5} - \frac{1}{2}$$

$$\therefore \quad 2\cos^2\frac{\pi}{5} - \cos\frac{\pi}{5} - \frac{1}{2} = 0 \text{ or equivalently } 4\cos^2\frac{\pi}{5} - 2\cos\frac{\pi}{5} - 1 = 0$$

iv

$$2\cos^2\frac{\pi}{5} - \cos\frac{\pi}{5} - \frac{1}{2} = 0$$

$$\therefore \quad 2\left(\cos^2\frac{\pi}{5} - \frac{1}{2}\cos\frac{\pi}{5} - \frac{1}{4}\right) = 0$$

$$\therefore \quad 2\left(\cos^2\frac{\pi}{5} - \frac{1}{2}\cos\frac{\pi}{5} + \frac{1}{16} - \frac{5}{16}\right) = 0$$

$$\therefore \quad 2\left(\left(\cos\frac{\pi}{5} - \frac{1}{4}\right)^2 - \frac{5}{16}\right) = 0$$

$$\therefore \quad 2\left(\cos\frac{\pi}{5} - \frac{1}{4}\right)^2 - \frac{5}{8} = 0$$

$$\therefore \quad 2\left(\cos\frac{\pi}{5} - \frac{1}{4}\right)^2 = \frac{5}{8}$$

$$\therefore \quad \left(\cos\frac{\pi}{5} - \frac{1}{4}\right)^2 = \frac{5}{16}$$

$$\therefore \quad \cos\frac{\pi}{5} - \frac{1}{4} = \pm\frac{\sqrt{5}}{4}$$

$$\therefore \quad \cos\frac{\pi}{5} = \frac{1}{4} \pm \frac{\sqrt{5}}{4}$$

$$\therefore \quad \cos\frac{\pi}{5} = \frac{1-\sqrt{5}}{4}, \ \frac{1+\sqrt{5}}{4}$$

but $\cos\frac{\pi}{5} > 0$, as $0 < \frac{\pi}{5} < \frac{\pi}{2}$

$$\therefore \quad \cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$$

5 a i

$$\text{LHS} = \cos\theta$$
$$= \cos\left(2 \times \frac{\theta}{2}\right)$$
$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$
$$\text{RHS} = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$$
$$= \frac{1 - \tan^2\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$$
$$= \cos^2\frac{\theta}{2}\left(1 - \tan^2\frac{\theta}{2}\right)$$
$$= \cos^2\frac{\theta}{2} - \frac{\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}$$
$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

Therefore $\text{LHS} = \text{RHS}$.

Hence $\cos\theta = \dfrac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$, as required.

ii

$$\text{RHS} = \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$$
$$= \frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$$
$$= \cos^2\frac{\theta}{2} \times 2\tan\frac{\theta}{2}$$
$$= \frac{2\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$
$$= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$= \sin\left(2 \times \frac{\theta}{2}\right)$$
$$= \sin\theta$$
$$= \text{LHS}$$

Hence $\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$, as required.

b

$$8\cos\theta - \sin\theta = 4$$

$$\therefore\quad 8\left(\frac{1-\tan^2\dfrac{\theta}{2}}{1+\tan^2\dfrac{\theta}{2}}\right) - \frac{2\tan\dfrac{\theta}{2}}{1+\tan^2\dfrac{\theta}{2}} = 4$$

$$\therefore\quad 8\left(1-\tan^2\frac{\theta}{2}\right) - 2\tan\frac{\theta}{2} = 4\left(1+\tan 2\frac{\theta}{2}\right)$$

$$\therefore\quad 8 - 8\tan^2\frac{\theta}{2} - 2\tan\frac{\theta}{2} = 4 + 4\tan^2\frac{\theta}{2}$$

$$\therefore\quad 12\tan^2\frac{\theta}{2} + 2\tan\frac{\theta}{2} - 4 = 0$$

$$\therefore\quad 6\tan^2\frac{\theta}{2} + \tan\frac{\theta}{2} - 2 = 0$$

$$\therefore\quad \left(3\tan\frac{\theta}{2} + 2\right)\left(2\tan\frac{\theta}{2} - 1\right) = 0$$

$$\therefore\quad 3\tan\frac{\theta}{2}+2 = 0 \text{ or } 2\tan\frac{\theta}{2} - 1 = 0$$

$$\therefore\quad \tan\frac{\theta}{2} = \frac{-2}{3} \quad \tan\frac{\theta}{2} = \frac{1}{2}$$